$$\lambda_{nm}^2 P_m{}^n (\cos\theta) - \delta \frac{dP_m{}^n (\cos\theta)}{d\theta} = 0 \quad \text{for } \theta = \pi/2$$

which together with (2.4) allows determination of the value m and λ_{nm}^2 (m is not an integer and $P_m{}^n(x)$ is the associated function of Legendre). For $\delta \ll \lambda^2$ approximate values of m are $m = n + k$ ($k = 1, 3, 5, .$)and therefore the application of the two different kinds of conditions to the free surface of fluid gives practically the same results.

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THE RELATION BETWEEN MATHEMATICAL EXPECTATIONS OF STRESS AND STRAIN TENSORS IN ELASTIC MICROHETEROGENEOUS MEDIA

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Microheterogeneous media (composite materials, polycrystals and others) are examined for which the elastic moduli tensor c_{ijmn} is considered a homogeneous random function of coordinates. The question of the relation between mathematical expectations of stresses $\langle\sigma_{ij}\rangle$ and strains $\langle\varepsilon_{ij}\rangle$ in such media was studied by a number of authors [1-5] under the condition that the fields of stresses and strains are statistically homogeneous. The author of [6] examined the case of inhomogeneous fields and proposed a method of solution for the inhomogeneous stochastic problem. In this paper the program outlined in [6] is carried out.

In Sect. 1 the initial stochastic inhomogeneous problem is reduced to an infinite sequence of homogeneous problems. This is achieved through the representation of the solution in the form of a series which satisfies the equilibrium equations for a volume element of the body, and the equations of compatibility of deformations. The coefficients of this series are homogeneous random tensor functions which are independent of body form and also independent of the determined external load acting on the body. These tensor functions depend only on the elastic properties of the body and are completely determined through the given random tensor c_{ijmn}.

In Sect. 2 the coefficients of the above mentioned series are expressed in terms

of the characteristics of the microstructure under the assumption that the medium is strongly isotropic. This assumption permits to avoid the limitation to the case of small inhomogeneities, and at the same time to avoid the need for giving multipoint correlation functions. It follows from the constructed solution that the dependence between $\langle\sigma_{ij}\rangle$and $\langle\varepsilon_{ij}\rangle$is analogous to the relation between stresses and strains in the multicouple stress theory of elasticity [7]. The transition is made from the differential to the integral form of the realtion. The integral form is characteristic for nonlocal theory of elasticity ([8] and others).

1. Let us examine a microheterogeneous medium for which at an arbitrary point the relation between stresses σ_{ij} and strains ε_{ij} obeys the generalized form of Hooke's law

$$\sigma_{ij} = c_{ijmn}\varepsilon_{mn} \tag{1.1}$$

Here the elastic moduli tensor $c_{ijmn}$is a random function of coordinates. It is assumed that the random fieldc $(\mathbf{r})$ $(\mathbf{r} = (x_1, x_2, x_3))$is homogeneous and ergodic. The tensors σ_{ij} and ε_{ij} will also be random functions, generally speaking inhomogeneous, because they depend not only on c $(\mathbf{r})$but also on the external load which in the following is assumed to be determined.

Let a characteristic volume of the microheterogeneous medium (a volume which in the structural sense is typical for the medium as a whole) be in equilibrium under the action of external forces. We represent the stress and strain tensors in the form

$$\sigma_{ij} = \langle\sigma_{ij}\rangle + \sigma_{ij}^*, \qquad \varepsilon_{ij} = \langle\varepsilon_{ij}\rangle + \varepsilon_{ij}^* \tag{1.2}$$

Here and in the following the angle brackets denote averaging over the entire representation. For the case of a statistically homogeneous quantity this averaging is identical to the average over the volume. Asterisks denote centered quantities. Tensors σ_{ij} and ε_{ij} must satisfy the equilibrium equations and the equations of compatibility of deformations.

$$\partial_j\sigma_{ij} + F_i = 0 \tag{1.3}$$

$$\epsilon_{ikm}\epsilon_{jnl}\partial_m\partial_n\varepsilon_{kl} = 0 \tag{1.4}$$

Here F_i is the determined vector of volume forces, ϵ_{imk} is the unitary alternative tensor of Levi-Civita. Substituting formulas (1.2) into Eqs. (1.3) and (1.4), we arrive at two systems of equations for the regular and fluctuating components of stress and strain tensors

$$\partial_j\langle\sigma_{ij}\rangle + F_i = 0, \qquad \epsilon_{imk}\epsilon_{jnl}\partial_k\partial_l\langle\varepsilon_{mn}\rangle = 0 \tag{1.5}$$

$$\partial_j\sigma_{ij}^* = 0, \qquad \epsilon_{imk}\epsilon_{jnl}\partial_k\partial_l\varepsilon_{mn}^* = 0 \tag{1.6}$$

Following the method proposed in [6], we shall look for a solution of system (1.6) in the form of the series

$$\varepsilon_{ij}^* = \overset{\circ}{\alpha}_{ijmn}\langle\varepsilon_{mn}\rangle + \sum_{\mu=1}^{\infty}\alpha^{\mu}_{ijmnp_1\ldots p_\mu}\partial_{p_1}\ldots\partial_{p_\mu}\langle\varepsilon_{mn}\rangle \tag{1.7}$$

Coefficients α°, α^μ are assumed to be homogeneous random functions of coordinates. Using (1.1) and (1.7), we have for the stress tensors

$$\sigma_{ij} = \overset{\circ}{A}_{ijmn}\langle\varepsilon_{mn}\rangle + \sum_{\mu=1}^{\infty}A^{\mu}_{ijmnp_1\ldots p_\mu}\partial_{p_1}\ldots\partial_{p_\mu}\langle\varepsilon_{mn}\rangle \tag{1.8}$$

Here

$$\overset{\circ}{A}_{ijmn} = c_{ijmn} + c_{ijkl}\overset{\circ}{\alpha}_{klmn}, \quad A^{\mu}_{ijmnp_1\ldots p_\mu} = c_{ijkl}\alpha^{\mu}_{klmnp_1\ldots r_\mu} \quad (\mu = 1, 2, \ldots) \tag{1.9}$$

Coefficients of series (1.8) are not centered quantities. In accordance with this we have

$$\langle\sigma_{ij}\rangle = \overset{\circ}{C}_{ijmn}\langle\varepsilon_{mn}\rangle + \sum_{\mu=1}^{\infty} C^{\mu}_{ijmnn_1\ldots p_\mu}\partial_{p_1}\cdots\partial_{p_\mu}\langle\varepsilon_{mn}\rangle \tag{1.10}$$

$$\sigma_{ij}^* = A^{(0)*}_{ijmn}\langle\varepsilon_{mn}\rangle + \sum_{\mu=1}^{\infty} A^{(\mu)*}_{ijmnp_1\ldots p_\mu}\partial_{p_1}\cdots\partial_{p_\mu}\langle\varepsilon_{mn}\rangle \tag{1.11}$$

$$\mathrm{C}^k = \langle\mathrm{A}^k\rangle \qquad (k = 0, 1, 2, \ldots)$$

Substituting series (1.7) and (1.11) into the system of equations (1.6), it is required that these equations be satisfied as a result of equality to zero of coefficients of tensor $\langle\varepsilon_{mn}\rangle$ and its partial derivatives with respect to coordinates. As a result we arrive at the recurrent sequence of systems of equations for determination of coefficients of series (1.7)

$$\partial_j A^{(0)*}_{ijmn} = 0, \qquad \epsilon_{irk}\epsilon_{jsl}\partial_k\partial_l\overset{\circ}{\alpha}_{rsmn} = 0 \tag{1.12}$$

$$\partial_j A^{(\mu)*}_{ijmn_1\ldots p_\mu} = - A^{(\mu-1)*}_{ip_\mu mnp_1\ldots r_{\mu-1}}\Big|_{(p_1\ldots p_\mu)}$$

$$\epsilon_{irk}\epsilon_{jsl}\partial_k\partial_l\alpha^{\mu}_{rsmnr_1\ldots p_\mu} = - \Psi^{\mu}_{ijmnr_1\ldots r_\mu}\Big|_{(p_1\ldots r_\mu)} \tag{1.13}$$

$$(\mu = 1, 2, \ldots)$$

$$\Psi^{\mu}_{ijmnp_1\ldots p_\mu} = \epsilon_{irk}\epsilon_{jsl}\left[2\delta_{p_\mu(k}\partial_{l)}\alpha^{(\mu-1)}_{rsmnp_1\ldots r_{\mu-1}} + \delta_{kp_\mu}\delta_{lr_{\mu-1}}\alpha^{(\mu-2)}_{rsmnp_1\ldots p_{\mu-2}}\right] \tag{1.14}$$

Here δ_{ij} is the Kronecker delta. Parentheses indicate symmetrization with respect to the corresponding indices.
In this manner we have instead of one stochastic inhomogeneous problem (1.6) an infinite sequence of homogeneous problems.

Each of the systems of equations (1.12), (1.13) is identical to equations of the theory of elasticity for a microheterogeneous medium with sources of internal stresses. The role of the strain tensor in such a medium is played by the tensor α^k ($k = 0, 1, 2, \ldots$)In system (1.12) the given quantity is represented by the random tensor c_{ijmn}. The right sides of each of the successive systems of equations are quantities which are determined in the solution of the previous equations. The requirement of homogeneity of tensors α^k (in some sense analogous to the requirement of periodicity of solution) in systems (1.12) and (1.13) replaces the boundary conditions. In this manner we can determine successively the coefficients of the series (1.7).

It is apparent that these coefficients are functions only of the elastic properties of the body and do not depend on the external load acting on the body. The external load enters into the series (1.7) by means of the determined tensor $\langle\varepsilon_{mn}\rangle$. This tensor is determined depending on the dimensions of the body and the forces acting on it by solving the system of equations (1.5).

2. Let us use the perturbation method [1-4] and rewrite the systems of equations (1.12), (1.13) in the following form:

$$\partial_j \langle c_{ijrs} \rangle \alpha^\circ_{rsmn} = -\partial_j a^{(\circ)*}_{ijmn}, \quad \epsilon_{irk}\epsilon_{jsl}\partial_k\partial_l \alpha^\circ_{rsmn} = 0 \tag{2.1}$$

$$\partial_j \langle c_{ijrs} \rangle \alpha^\mu_{rsmnp_1\ldots p_\mu} = -\partial_j a^{(\mu)*}_{ijmnp_1\ldots p_\mu} - \langle c_{ip_\mu rs} \rangle \alpha^{(\mu-1)}_{rsmnp_1\ldots p_{\mu-1}} - \tag{2.2}$$

$$- a^{(\mu-1)*}_{ip_\mu mnp_1\ldots p_{\mu-1}} \Big| (p_1 \ldots p_\mu)$$

$$\epsilon_{irk}\epsilon_{jsl}\partial_k\partial_l \alpha^\mu_{rsmnp_1\ldots p_\mu} = -\Psi^\mu_{ijmnp_1\ldots p_\mu} \Big| (p_1 \ldots p_\mu)$$

Here

$$a^\circ_{ijmn} = A^\circ_{ijmn} - \langle c_{ijrs} \rangle \alpha^\circ_{rsmn} \tag{2.3}$$

$$a^\mu_{ijmnp_1\ldots p_\mu} = A^{\mu(}_{ijmnp_1\ldots p_\mu} - \langle c_{ijrs} \rangle \alpha^\mu_{rsmnp_1\ldots p_\mu}$$

$$(\mu = 1, 2, \ldots)$$

For later use we need only the particular solutions of systems of equations (2.1), (2.2). These solutions are subject to the following additional stochastic condition: $\langle \alpha^k \rangle = 0$ ($k = 0,1,2\ldots$). Through a direct verification it is easy to become convinced that the expressions

$$\alpha^\circ_{klmn}(\mathbf{r}) = \int_V U_{i(k,l)j}(\rho)\, a^{(0)*}_{ijmn}(\mathbf{r}')\, dV' \qquad \rho = \mathbf{r}' - \mathbf{r} \qquad (\mu = 1, 2, \ldots)$$

$$\alpha^\mu_{klmnp_1\ldots p_\mu}(\mathbf{r}) = \Phi^\mu_{klmnp_1\ldots p_\mu}(\mathbf{r}) + \int_V U_{i(k,l)j}(\rho)\, a^{(\mu)*}_{ijmnp_1\ldots p_\mu}(\mathbf{r}')\, dV' \tag{2.4}$$

transform the equations of system (2.1), (2.2) into identities and that they satisfy the mentioned condition. In (2.4) we have

$$\Phi^\mu_{klmnp_1\ldots p_\mu}(\mathbf{r}) = \frac{1}{\mu!}\int_V Q^\mu_{ijklp_1\ldots p_\mu}(\rho)\, c^*_{ijmn}(\mathbf{r}')\, dV' + \int_V U_{i(k,l)}(\rho)\, a^{(\mu-1)*}_{ip_\mu mnp_1\ldots p_{\mu-1}}(\mathbf{r}')\, dV'$$

$$Q^\mu_{ijklp_1\ldots p_\mu}(\mathbf{r}) = U_{i(k,l)j}(\mathbf{r})\, x_{p_1} \ldots x_{p_\mu} \qquad (\mu = 1, 2, \ldots) \tag{2.5}$$

Here $U_{kl}(\mathbf{r})$ is Green's tensor for a homogeneous medium with elastic moduli $\langle c_{ijkl} \rangle$, which are solutions of the following equation:

$$\langle c_{ijkl} \rangle \partial_i \partial_k U_{ln}(\rho) = -\delta_{jn}\delta(\rho) \tag{2.6}$$

where $\delta(r)$ is the Dirac delta function. In particular, if the medium is macroscopically isotropic, then

$$\langle c_{ijkl} \rangle = \bar\lambda \delta_{ij}\delta_{kl} + 2\bar\mu I_{ijkl}$$

$$U_{kl}(\mathbf{r}) = \frac{1}{4\pi\bar\mu}\left(\frac{\delta_{kl}}{r} - \frac{1}{2} g \frac{\partial^2 r}{\partial x_k \partial x_l}\right) \qquad r = |\mathbf{r}| \tag{2.7}$$

$$g = \frac{\bar\lambda' + \bar\mu}{\bar\lambda + 2\bar\mu}, \qquad I_{ijkl} = \delta_{i(k}\delta_{j)l}$$

Expressions (2.4) represent integro-differential equations in α^k and can be solved by the iteration method. However, we do not need the functions $\alpha^k(r)$ themselves, but the mathematical expectations of their convolution with the tensor of fluctuations of elastic moduli $\langle c^*\alpha^k \rangle$ ($k = 0,1,2,\ldots$). These expressions can be found with the aid of Eqs. (2.4) and the iteration process proposed in [4].

In the following we shall limit ourselves only to the examination of macroscopically isotropic microheterogeneous media. Furthermore, it will be assumed that the medium is strongly isotropic [3, 4]. The last statement means that single-point correlation functions for c^*_{ijmn} depend only on the distance between the points of the field, i.e.

$$\langle \mathbf{c}^*(\mathbf{r})\, \mathbf{c}^*(\mathbf{r}) \ldots \mathbf{c}^*(\mathbf{r}')\rangle = \mathbf{C}(\rho) \tag{2.8}$$

In addition, in papers [3, 4] it was assumed

$$\langle \mathbf{c}^*(\mathbf{r})\, \mathbf{c}^*(\mathbf{r}) \ldots \mathbf{a}^{(0)*}(\mathbf{r}')\rangle = \mathbf{B}^\circ(\rho) \tag{2.9}$$

It is natural to supplement the condition of strong isotropy by the following requirement:

$$\langle \mathbf{c}^*(\mathbf{r})\, \mathbf{c}^*(\mathbf{r}) \ldots \mathbf{a}^{(\mu)*}(\mathbf{r}')\rangle = \mathbf{B}^\mu(\rho) \tag{2.10}$$

But then

$$\mathbf{B}^\mu(\rho) = 0 \qquad (\mu = 1, 3, 5, \ldots) \tag{2.11}$$

as tensors of odd valency. Conditions (2.8) - (2.11) permit to carry out the calculations by the same method as in paper [4].

Let us multiply the right and left parts of Eqs. (2.4) by $c^*_{ijkl}(\mathbf{r})$ and compute the mathematical expectations of both parts

$$\langle \overset{*}{c}_{ijkl} \overset{\circ}{\alpha}_{klmn}\rangle = \int_V U_{\lambda(k,l)\nu}(\rho)\, B^{(\circ)\,ijkl}_{\lambda\nu mn}(\rho)\, dV \tag{2.12}$$

$$\langle \overset{*}{c}_{ijkl} \alpha^{\mu}_{klmnp_1\ldots p_\mu}\rangle = \int_V Q_{\lambda\nu klp_1\ldots p_\mu}(\rho)\, C^{ijkl}_{\lambda\nu mn}(\rho)\, dV +$$
$$+ \int_V U_{\lambda(k,l)\nu}(\rho)\, \mathbf{B}^{(\mu)\,ijkl}_{\lambda\nu mnp_1\ldots p_\mu}(\rho)\, dV \qquad (\mu = 2, 4, \ldots)$$

The second derivative of the Green function is

$$U_{\lambda(k,l)\nu}(\mathbf{r}) = -E_{kl\lambda\nu}\delta(\mathbf{r}) + G_{kl\lambda\nu}(\mathbf{r}) \tag{2.13}$$

Here

$$E_{kl\lambda\nu} = \frac{\alpha}{9\bar{k}}\delta_{kl}\delta_{\lambda\nu} + \frac{\beta}{2\bar{\mu}}\left(I_{kl\lambda\nu} - \frac{1}{3}\delta_{kl}\delta_{\lambda\nu}\right) \tag{2.14}$$

$$\alpha = 3 - 5\beta = \bar{k} \Big/ \left(\bar{k} + \frac{4}{3}\bar{\mu}\right)', \quad \bar{k} = \bar{\lambda} + \frac{2}{3}\bar{\mu}$$

$$G_{kl\lambda\nu}(\mathbf{r}) = \frac{1}{4\pi\bar{\mu}r^3}\Big\{\delta_{\lambda k}(3\psi_{l\nu} - \delta_{l\nu}) - \frac{1}{2}g\,[-\delta_{kl}\delta_{\lambda\nu} - 2I_{kl\lambda\nu} +$$
$$+ 18\delta_{(kl}\psi_{\lambda\nu)} - 15\psi_{k\nu}\psi_{\lambda\nu}]\Big\}_{(kl)}, \quad \psi_{kl} = \frac{x_k x_l}{r^2} \tag{2.15}$$
$$6\delta_{(kl}\psi_{\lambda\nu)} = \delta_{kl}\psi_{\lambda\nu} + \delta_{k\nu}\psi_{l\lambda} + \delta_{k\lambda}\psi_{l\nu} + \delta_{l\lambda}\psi_{k\nu} + \delta_{l\nu}\psi_{k\lambda} + \delta_{\lambda\nu}\psi_{kl}$$

We note that

$$\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\psi G_{kl\lambda\nu}(r, \theta, \psi) \equiv 0 \tag{2.16}$$

where r, θ, ψ are spherical coordinates in the $\mathbf{r}$-space. The correlation function (2.8) can be represented in the form [9]

$$\mathbf{C}(\rho) = \mathbf{C}(0)\,\varphi(\rho) \tag{2.17}$$

where $\mathbf{C}(0)$ and $\varphi(\rho)$ are tensor and coordinate dependences of the correlation function $\mathbf{C}(\rho)$, and $\varphi(0) = 1$, $\varphi(\infty) = 0$. Taking into account (2.13) - (2.17), we integrate (2.12) and by changing to a system of notation which is independent of coordinates we obtain

$$\langle c^* \alpha^k \rangle = \langle c^* P^k c^* \rangle - \langle c^* E c^* \alpha^k \rangle \qquad (k = 0, 2, 4, \ldots) \tag{2.18}$$

$$P^\circ = -E \tag{2.19}$$

$$P^\mu = (P^\mu_{kl\lambda\nu p_1 \ldots p_\mu}) = \int\limits_V G_{kl\lambda\nu}(r) x_{p_1} \ldots x_{p_\mu} \varphi(r)\, dV$$

The expression $\langle c^* E c^* \alpha^k \rangle$, which enters into the right side of (2.18) can be computed with the aid of Eqs. (2.12) and the conditions of strong isotropy

$$\langle c^* E c^* \alpha^k \rangle = \langle c^* E c^* P^k c^* \rangle - \langle (c^* E)^2 c^* \alpha^k \rangle + \langle (c^* E)^2 \rangle \langle c^* \alpha^k \rangle$$

Repeating the process, we find

$$\langle (c^* E)^{n-1} c^* \alpha^k \rangle = \langle c^* (E c^*)^{n-1} P^k c^* \rangle - \langle (c^* E)^n c^* \alpha^k \rangle + \langle (c^* E)^n \rangle \langle c^* \alpha^k \rangle$$
$$(n = 1, 2, \ldots)$$

From this we arrive at the following equation:

$$\langle c^* \alpha^k \rangle = L^k - L^\circ E \langle c^* \alpha^k \rangle \tag{2.20}$$

Here

$$L^k = \sum_{n=0}^{\infty} (-1)^n \langle c^* (E c^*)^n P^k c^* \rangle = \langle c^* (I + E c^*)^{-1} P^k c^* \rangle \tag{2.21}$$

Solving Eq. (2.20) with respect to $\langle c^* \alpha^k \rangle$, we finally find

$$\langle c^* \alpha^k \rangle = (I + L^\circ E)^{-1} L^k \qquad (k = 0, 2, 4, \ldots) \tag{2.22}$$

According to (1.10) we can now write

$$\langle \sigma \rangle = C^\circ \langle \varepsilon \rangle + \sum_\mu c^\mu \underbrace{\nabla \ldots \nabla}_{} \langle \varepsilon \rangle \qquad (\mu = 2, 4, \ldots) \tag{2.23}$$

Here

$$C^\circ = \langle c \rangle + (I + L^\circ E)^{-1} L^\circ, \qquad C^\mu = (I + L^\circ E)^{-1} L^\mu \tag{2.24}$$

If the volume which is being examined is in a macroscopically homogeneous deformed state, i.e. $\langle \varepsilon \rangle =$ const, then

$$\langle \sigma \rangle = C^\circ \langle \varepsilon \rangle \tag{2.25}$$

In this connection C° is the tensor of macroscopic elastic moduli of a microheterogeneous medium which is homogeneously deformed. An expression for C° in (2.24) was obtained in [4].

We can show that the differential operator of infinitely high order in (2.23) is equivalent to the integral operator, i.e.

$$\langle \sigma \rangle = C^\circ \langle \varepsilon \rangle + \int\limits_V C(r - r') \langle \varepsilon(r') \rangle \, dV' \tag{2.26}$$

In this connection the kernel of this operator is expressed through the microelastic properties of the material in the following manner

$$C(r) = (I + L^\circ E)^{-1} L(r), \qquad L(r) = \langle c^* (I + c^* E)^{-1} H(r) c^* \rangle$$
$$H(r) = (H_{ijkl}(r)) = G_{ijkl}(r) \varphi(r) \tag{2.27}$$

In fact, representing (2.26) in the form

$$\langle \sigma \rangle = C^\circ \langle \varepsilon \rangle + \int\limits_V C(\rho) \langle \varepsilon(r + \rho) \rangle \, d\rho$$

and expanding $\langle\varepsilon\ (r+\rho)\rangle$ in a Taylor series in the vicinity of the point r

$$\langle\varepsilon_{mn}(\mathbf{r}+\rho)\rangle = \langle\varepsilon_{mn}\rangle + \sum_{\mu=1}^{\infty}\frac{1}{\mu!}\frac{\partial^{\mu}\langle\varepsilon_{mn}\rangle}{\partial x_{p_1}\ldots\partial x_{p_\mu}}\xi_{p_1}\ldots\xi_{p_\mu}$$

$$\xi_p = x_p' - x_p$$

we obtain expression (2.23). If the inhomogeneity is small [1], then in series (2.21) terms having an order higher than $(c^*)^2$ can be neglected. Equations (2.24) and (2.27) are simplified

$$C^*_{ijmn} = \langle c_{ijmn}\rangle - E_{klrs}C^{ijkl}_{rsmn} \tag{2.28}$$

$$C^{\mu}_{ijmnp_1\ldots,p_\mu} = P^{\mu}_{klrsp_1\ldots p_\mu}C^{ijkl}_{rsmn}\quad(\mu=2,4,\ldots)\quad C_{ijmn} = C^{ijkl}_{rsmn}G_{klrs}(\mathbf{r})\,\varphi(r) \tag{2.29}$$

Expression (2.28) was obtained in [1]. We note that for derivation of Eqs. (2.28) and (2.29) there is no need for assumptions (2.8) - (2.11).

In this manner, giving up the requirement of macroscopic homogeneity for the deformation field ($\langle\varepsilon\rangle \neq$ const) leads to the nonlocal relation between mathematical expectations of stress and strain tensors (2.23) or (2.26). This nonlocal character is on one hand due to the inhomogeneity of average deformations in the material, and on the other hand due to the existence of the correlation for the field of fluctuations of elastic moduli. The nonlocal character disappears if $\langle\varepsilon\rangle$ = const by virtue of (2.8), (2.16); it also disappears in the absence of correlation, i.e. if $\varphi(r) = 0$ for $r \neq 0$ ("perfectly disordered" model of a microheterogeneous medium according to Kröner [2]). If it is assumed that the inhomogeneity of the material is connected with the presence of determined boundaries of grain separation, i.e. in the transition from grain to grain the properties of the medium change in a jump, then the spatial part of the correlation function $\varphi(r)$ can be selected in the form [9]

$$\varphi(r) = e^{-r/a} \tag{2.30}$$

Here a is the scale of correlation. The kernel of the integral operator in (2.26) is then completely determined by given quantities. The defining relation (2.26) solves the problem of connection between average stresses and strains in the material. The tensors $\langle\sigma_{ij}\rangle$ and $\langle\varepsilon_{ij}\rangle$ themselves must be determined from the solution of the system of equations (1.5) supplemented by the usual boundary conditions

$$\langle\sigma_{ij}\rangle n_j = f_i$$

where f_i is the determined vector of surface forces. Substituting the defining relation (2.26) into Eq. (1.5) and taking into account that $\langle\varepsilon_{ij}\rangle = {}^1/_2\,(\partial_j\langle u_j\rangle + \partial_j\langle u_i\rangle)$ we obtain a system of three integro-differential equations for the determination of vector $\langle u_i\rangle$.

An approximate formulation of the problem consisting in the following is of interest: the integral operator in (2.26) is replaced by the differential one (2.23) where a finite number of terms are retained. Such a substitution leads to a system of differential equations with respect to $\langle u_i\rangle$. These equations have the order $2(1+\mu)$ where μ is the number of terms retained in (2.23). However, the problem of approximate formulation of boundary value problems remains open and requires additional investigation.

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REPRESENTATION OF THE RESOLVENTS OF OPERATORS OF VISCOELASTICITY IN TERMS OF SPECTRAL DISTRIBUTION FUNCTIONS

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A general representation of the resolvent in terms of a reduced distribution function of a viscoelastic spectrum of the initial kernel is obtained. The method is illustrated using the widely known operators of viscoelasticity. Resolvents are constructed for the generalized fractional exponential kernel and the logarithmic kernels.

Viscoelastic behavior of the real bodies (polymers in particular) can also be described [1 - 3] by other equations containing the Volterra operator

$$P^*(\cdots) = \int_0 P(t-\tau)(\cdots)\,d\tau \tag{0.1}$$

where $P(t)$ is the operator kernel. If $I - \varkappa P^*, -\infty < \varkappa < \infty$ is a complete operator of viscoelasticity, then the operator